The Chinese University of Hong Kong
Department of Mathematics
MMAT 5340 Homework 10 (No submission required)

1. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with a state space $S = \{1, 2, 3\}$ and the transition matrix

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix}$$

   (a) Show that the Markov chain is irreducible and recurrent.

   (b) Find the stationary distribution π such that $\pi^T P = \pi^T$.

   (c) (Not required for Final exam) Let $f(x) = x$, compute

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(X_k)}{n}.$$

Solution

   (a) Since $P(1, 2) = 0.1 = P(2, 1) = 0.1$ and $P(2, 3) = P(3, 2) = 0.1$, then $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$. We can conclude that any two states in this chain are intercommunicate, so it is irreducible. Since this chain is finite and irreducible, then it is recurrent.

   (b) Let $\pi = \pi = (\pi(1), \pi(2), \pi(3))$. Solving $\pi^T P = \pi^T$ is equivalent to solve

$$\begin{bmatrix} \pi(1) & \pi(2) & \pi(3) \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix} = \begin{bmatrix} \pi(1) & \pi(2) & \pi(3) \end{bmatrix},$$

   which implies

$$0.9\pi(1) + 0.1\pi(2) = \pi(1)$$

$$0.1\pi(1) + 0.8\pi(2) + 0.1\pi(3) = \pi(2)$$

$$0.1\pi(1) + 0.9\pi(3) = \pi(3)$$

   It follows that $\pi(1) = \pi(2) = \pi(3)$, together with $\pi(1) + \pi(2) = \pi(3) = 1$ yields $\pi(1) = \pi(2) = \pi(3) = \frac{1}{3}$.

   (c)

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(X_k)}{n} = \sum_{x\in S} \pi(x) f(x) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{1}{3} \cdot 3 = 2.$$

2. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with a state space $S = \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and

$$P(x, x + 1) = p, \quad P(x, 0) = 1 - p$$

   for some $p \in (0, 1)$.

(a) Find the transition matrix P of X.

(b) Is this Markov chain irreducible or reducible?

Recall that $\tau_x^1 := \inf\{n \geq 1 : X_n = x\}$.

(c) Show that $P_x[\tau_0^1 = n] := P[\tau_0^1 = n|X_0 = x] = p^{n-1}(1-p)$. Compute $\mathbb{E}_x[\tau_0^1]$.

(d) Prove that for $x \geq 2$

$$\mathbb{E}_0[\tau_x^1] = \mathbb{E}_0[\tau_{x-1}^1] + \mathbb{E}_{x-1}[\tau_x^1]$$

and

$$\mathbb{E}_{x-1}[\tau_x^1] = 1 + (1-p)\mathbb{E}_0[\tau_x^1].$$

Hint: For the first equality, first show that for $\forall m, n \geq 1$

$$\mathbb{P}_0[\tau_{x-1}^1 = m, \tau_x^1 - \tau_{x-1}^1 = n] = \mathbb{P}_0[\tau_{x-1}^1 = m] \cdot \mathbb{P}_{x-1}[\tau_x^1 = n].$$

Then use the fact that

$$\mathbb{P}_0[\tau_x^1 - \tau_{x-1}^1 = n] = \sum_{m=1}^{\infty} \mathbb{P}_0[\tau_{x-1}^1 = m] \cdot \mathbb{P}_{x-1}[\tau_x^1 = n].$$

For the second equality, you may accept that

$$\mathbb{P}_{x-1}[\tau_x^1 = n] = p \cdot \mathbf{1}_{\{n=1\}} + (1-p) \cdot \mathbf{1}_{\{n \geq 1\}} \cdot \mathbb{P}_0[\tau_x^1 = n-1].$$

(e) Optional. Define

$$f(x) := \mathbb{E}_0[\tau_x^1], \quad g(x) := \mathbb{E}_x[\tau_x^1]$$

Deduce that

$$f(x) = (\frac{1}{p})^x \frac{1}{1-p} - \frac{1}{1-p} \quad \text{for} x \geq 1$$

and

$$g(x) = (\frac{1}{p})^x \frac{1}{1-p}$$

Hint: Compute $f(1)$. Define $u(x) := f(x) + \frac{1}{1-p}$, then use the result from (d) to deduce $u(x) = \frac{u(x-1)}{p}$. Note that $g(x) = \mathbb{E}_0[\tau_x^1] + \mathbb{E}_x[\tau_0^1]$.

(f) Check that

$$\pi(x) = \frac{1}{g(x)} = \frac{1}{\mathbb{E}_x[\tau_x^1]}$$

is a stationary distribution. Find the stationary probability for $x \in S$.

Solution

(a)

$$P = \begin{bmatrix} 1-p & p & 0 & \cdots & 0 & \cdots \\ 1-p & 0 & p & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-p & 0 & 0 & \cdots & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

(b) Since $P(x,0) = 1-p > 0$ for $\forall x \in S$, then $x \to 0$.
Letting 0 goes x by $0 \to 1 \to 2 \to \cdots \to x-1 \to x$ with probability $p^x > 0$. Hence $0 \leftrightarrow x$ and $i \leftrightarrow j$ for any $i, j \in S$, which implies that the chain is irreducible.

(c)

$$\begin{aligned} P_x[\tau_0^1 = n] &:= P[\tau_0^1 = n | X_0 = x] \\ &= P(X_n = 0, X_{n-1} \neq 0, X_{n-1} \neq 0, \cdots, X_2 \neq 0, X_1 \neq 0 | X_0 = x) \\ &= (1-p)p^{n-1} \end{aligned}$$

$$\begin{aligned} \mathbb{E}_x[\tau_0^1] &= \sum_{n=1}^{\infty} n(1-p)p^{n-1} \\ &= (1-p)\frac{d(\sum_{n=1}^{\infty} p^n)}{dp} \\ &= (1-p)\frac{d(\frac{1}{1-p}-1)}{dp} \\ &= (1-p)\frac{1}{(1-p)^2} \\ &= \frac{1}{(1-p)} \end{aligned}$$

(d) For $\forall m, n \geq 1$, we have

$$\mathbb{P}_0[\tau_{x-1}^1 = m, \tau_x^1 - \tau_{x-1}^1 = n] = \mathbb{P}_0[\tau_{x-1}^1 = m] \cdot \mathbb{P}_0[\tau_x^1 - \tau_{x-1}^1 = n | \tau_{x-1}^n = m]. \qquad (1)$$

Since

$$\begin{aligned} &\mathbb{P}_0[\tau_x^1 - \tau_{x-1}^1 = n | \tau_{x-1}^n = m] \\ &= \mathbb{P}_0[X_{m+1} \neq x, \cdots, X_{m+n-1} \neq x, X_{m+n} = x | X_0 \neq x-1, \cdots, X_{m-1} \neq x-1, X_m = x-1] \\ &= \mathbb{P}_0[X_{m+1} \neq x, \cdots, X_{m+n-1} \neq x, X_{m+n} = x | X_m = x-1] \\ &= \mathbb{P}_{x-1}[X_1 \neq x, \cdots, X_{n-1} \neq x, X_n = x] \\ &= \mathbb{P}_{x-1}[\tau_x^1 = n] \end{aligned}$$

then equation(1) becomes

$$\mathbb{P}_0[\tau_{x-1}^1 = m, \tau_x^1 - \tau_{x-1}^1 = n] = \mathbb{P}_0[\tau_{x-1}^1 = m] \cdot \mathbb{P}_{x-1}[\tau_x^1 = n].$$

Hence

$$\begin{aligned} \mathbb{P}_0[\tau_x^1 - \tau_{x-1}^1 = n] &= \sum_{m \geq 1} \mathbb{P}_0[\tau_{x-1}^1 = m, \tau_x^1 - \tau_{x-1}^1 = n] \\ &= \sum_{m \geq 1} \mathbb{P}_0[\tau_{x-1}^1 = m] \cdot \mathbb{P}_{x-1}[\tau_x^1 = n] \\ &= \mathbb{P}_{x-1}[\tau_x^1 = n] \end{aligned}$$

it follows that

$$\mathbb{E}_0[\tau_x^1 - \tau_{x-1}^1] = \mathbb{E}_{x-1}[\tau_x^1]$$

and

$$\mathbb{E}_0[\tau_x^1] = \mathbb{E}_0[\tau_{x-1}^1] + \mathbb{E}_{x-1}[\tau_x^1].$$

Then we prove the second formula. We consider

$$\begin{aligned}\mathbb{P}_{x-1}[\tau_x^1 = n] &= \mathbb{P}[\tau_x^1 = n | X_0 = x-1] \\ &= \mathbb{P}[\tau_x^1 = n | X_1 = x] \cdot \mathbb{P}[X_1 = x | X_0 = x-1] \\ &+ \mathbb{P}[\tau_x^1 = n | X_1 = 0] \cdot \mathbb{P}[X_1 = 0 | X_0 = x-1]\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}_{x-1}[\tau_x^1] &= \sum_{n\geq 1} n\mathbb{P}_{x-1}[\tau_x^1 = n] \\ &= p + \sum_{n=2}^{\infty} n(1-p)\mathbb{P}_0[\tau_x^1 = n-1] \\ &= p + \sum_{n=2}^{\infty}(1+n-1)(1-p)\mathbb{P}_0[\tau_x^1 = n-1] \\ &= p + (1-p) + \sum_{n=2}^{\infty}(n-1)(1-p)\mathbb{P}_0[\tau_x^1 = n-1] \\ &= 1 + \sum_{n=1}^{\infty} n(1-p)\mathbb{P}_0[\tau_x^1 = n] \\ &= 1 + (1-p)\mathbb{E}_0[\tau_x^1]\end{aligned}$$

(e) Since

$$\mathbb{P}_0[\tau_1^1 = n] = \mathbb{P}_0[X_0 = 0, \cdots, X_{n-1} = 0, X_n = 1] = p(1-p)^{n-1}$$

then

$$f(1) = \mathbb{E}_0[\tau_1^1] = \sum_{n=1}^{\infty} np(1-p)^{n-1} = -p(\sum_{n=1}^{\infty}(1-p)^n)' = \frac{1}{p}$$

Set $u(x) = f(x) + \frac{1}{1-p}$. Note that

$$f(x) = f(x-1) + 1 + (1-p)f(x)$$

then

$$f(x) = \frac{1}{p}(f(x-1)+1)$$

and

$$\begin{aligned}u(x) &= \frac{u(x-1)}{p} = (\frac{1}{p})^{x-1}u(1) = (\frac{1}{p})^{x-1}[f(1) + \frac{1}{1-p}] \\ &= (\frac{1}{p})^{x-1}[\frac{1}{p} + \frac{1}{1-p}] = (\frac{1}{p})^x \frac{1}{1-p}\end{aligned}$$

Hence

$$f(x) = u(x) - \frac{1}{1-p} = (\frac{1}{p})^x \frac{1}{1-p} - \frac{1}{1-p}.$$

For $x \geq 1$

$$\begin{aligned}g(x) := \mathbb{E}_x[\tau_x^1] &= \mathbb{E}_x[\tau_0^1] + \mathbb{E}_0[\tau_x^1] \\ &= \frac{1}{1-p} + (\frac{1}{p})^x \frac{1}{1-p} - \frac{1}{1-p} \\ &= (\frac{1}{p})^x \frac{1}{1-p}\end{aligned}$$

For $x = 1$, $\mathbb{E}_0[\tau_0^1] = \frac{1}{1-p}$. Hnece $g(x) = (\frac{1}{p})^x \frac{1}{1-p}$ for all $x \in S$.

(f) By applying (e), we have

$$\pi(x) = p^x(1-p), \quad p\pi(x) = \pi(x+1)$$

and

$$\sum_{n=1}^{\infty} \pi(n) = (1-p)\sum_{n=1}^{\infty} p^n = (1-p)(\frac{1}{1-p} - 1).$$

Hence

$$\sum_{n=0}^{\infty} \pi(n)(1-p) = (1-p)^2 + \sum_{n=1}^{\infty} \pi(n)(1-p) = (1-p)^2 + (1-p)\sum_{n=1}^{\infty} \pi(n) = \pi(0)$$

Then

$$\begin{bmatrix}\pi(0) & \pi(1) & \cdots & \pi(n) & \cdots\end{bmatrix} \begin{bmatrix} 1-p & p & 0 & \cdots & 0 & \cdots \\ 1-p & 0 & p & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-p & 0 & 0 & \cdots & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix}\pi(0) & \pi(1) & \cdots & \pi(n) & \cdots\end{bmatrix}.$$

Therefor $\pi(x)$ is a stationary distribution.